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On the departure of a sphere from contact with a permeable membrane

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SUMMARY

A sphere in contact with a porous membrane can depart with non-zero velocity under the influence of a finite force. The flow field and the viscous force resisting the motion are evaluated by reducing the Stokes equations to a fourth-order ordinary differential equation through the use of conformal mapping. Explicit results are given for high and low membrane permeabilities by using regular and singular perturbation techniques respectively.

1. Introduction

The motion of particles near porous surfaces is a phenomenon characteristic to separation processes. In many processes in engineering systems as well as in biological organs the motion of a particle near a surface, through which fluid can penetrate, is the most important limiting step. Usually the macroscopic particle is small enough for inertial effects to be negligible yet large enough to be prevented from passing through the microscopic pores in the surface. This is equally true during filtration of suspensions or in the prevention of passage of macromolecules through cell membranes.

Problems concerning the motion of a solid sphere near an impermeable solid surface were intensively treated in the past (Stimson and Jeffery [1], Brenner [2] and a series of papers by O'Neill, e.g. [3]). A common feature to these solutions is the result that the closer the particle to the immobile surface the larger the force acting on it. The latter becomes infinite when contact occurs at non-zero velocities. Conversely, a sphere can approach or depart from a solid surface under the action of a finite force only at zero velocity. This characteristic stems from the requirement that the velocity field be single-valued at the point of contact, and that no-slip conditions prevail there. Recently, it has been suggested that conditions at the surface of a porous material contain penetration of fluid as well as tangential slip (Beavers & Joseph [4], Saffman [5]). Clearly, both changes in the conditions may result in fundamental differences in the kinematics and the dynamics of the sphere near the surface. In particular, non-zero departure velocities under the influence of a finite force can be expected.

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Indeed, Goren [6] has recently solved for the force acting on a solid sphere approaching a thin permeable membrane and found that the force on the sphere remains finite as the separation gap diminishes. Goren observed that maximum values for the force were not necessarily obtained at contact, depending on the magnitude of the membrane permeability. Unfortunately, the procedure which he used to obtain solutions is not valid at contact and definite values for the forces there still await exact evaluation.

In this communication, we solve for the flow field and the force acting on a solid sphere slowly departing from contact with a thin permeable membrane. It is assumed that the length scale associated with the pore sizes is much smaller than any macroscopic scale in the field. Furthermore, no flow occurs in the lateral direction within the membrane. The sphere is impermeable. In Section 2 the general problem is stated. In Section 3 we describe the use of conformal mapping to reduce the Stokes problem to a fourth-order differential equation in the continuous eigenvalue domain. The interesting functionals are discussed in Section 4, while perturbation methods are employed in Sections 5 and 6 to obtain exact solutions in asymptotic cases.

2. The general equations

Consider a solid sphere of radius a touching a horizontal porous thin membrane. The two half-spaces, below and above the membrane, are filled with viscous fluid having a viscosity μ. The sphere is departing from the membrane in a direction perpendicular to it at a velocity V. Since the membrane is immobile but permeable to fluid, motion will be induced on both of its sides. We further neglect inertia effects in the entire space. The inertialess motion of the fluid below and above the membrane obeys the Stokes equations which, in view of the axial symmetry, are of the form

$$E^4 \Psi^{\mathrm{I}} = E^4 \Psi^{\mathrm{II}} = 0 . \tag{1}$$

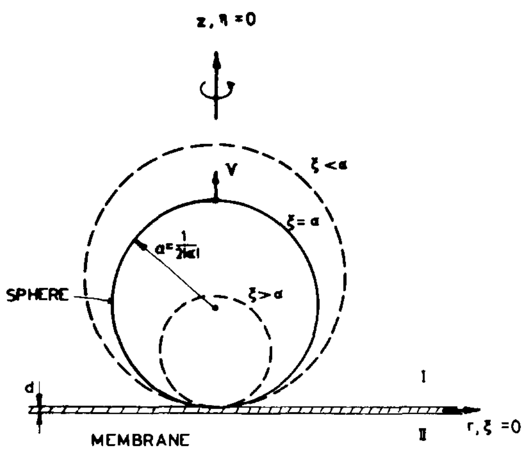

Figure 1 A sphere in contact with a permeable membrane

Here Ψ^{I} and Ψ^{II} are the stream functions in the upper and lower half-spaces respectively, and

$$E^2 \equiv r \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right) + \frac{d^2}{dz^2} \tag{2}$$

where r and z are cylindrical coordinates as depicted in Figure 1. For further discussion, we shall use non-dimensional variables with distances normalized by a, velocities by V, stream-functions by Va^2 and pressure by $\mu V/a$.

On the surface of the sphere no-slip conditions prevail,

$$u_z = 1, \quad u_r = 0, \tag{3}$$

while all velocity components decay at infinity.

The conditions at the membrane which is assumed to occupy the plane $z = 0$ deserve some attention. General boundary conditions for Stokes flow over porous surfaces were recently employed by Jones [7] and Nir [8], following Beavers & Joseph [4]. Here, for the sake of mathematical simplicity, we shall assume no tangential slip on both sides of the solid membrane, hence

$$u_r^{\mathrm{I}} = u_r^{\mathrm{II}} = 0 \quad \text{at} \quad z = 0 \tag{4}$$

with I and II referring to the upper and lower sides. The continuity of pressure and normal velocity take the forms

$$u_z^{\mathrm{I}} = u_z^{\mathrm{II}} \tag{5}$$

and

$$u_z^{\mathrm{I,II}} = -\beta(p^{\mathrm{I}} - p^{\mathrm{II}}) \tag{6}$$

where p is the pressure; β is a non-dimensional coefficient, $\beta = \kappa/(ad)$, with κ being the membrane permeability and d its thickness. Note that condition (6) is valid only for unidirectional flow within the membrane. This particular choice of conditions (4)-(6) will enable a direct comparison with the work of Goren [6].

The solution of (1) subject to conditions (3)-(6) and the requirement that the velocities vanish at infinity is facilitated by the use of the mapping

$$z + ir = \frac{i}{\eta + i\xi}. \tag{7}$$

Stokes flows for similar geometries were successfully solved using these so-called tangent-sphere coordinates ([9],[10],[11]). Surfaces of constant ξ describe spheres of radius $(2|\xi|)^{-1}$, all tangent to the surface $z = 0$ ($\xi = 0$) in which the thin membrane is assumed to be lying (see Figure 1). The coordinates of a point in space are given by

$$z = \frac{\eta}{\xi^2 + \eta^2}, \quad r = \frac{\eta}{\xi^2 + \eta^2} \tag{8}$$

where $\xi^2 + \eta^2$ is the metrical. We denote the surface of the sphere of radius unity by α with $\alpha = \frac{1}{2}$.

In terms of the mapping coordinates the operator E^2 has the form

$$E^2 = \eta(\xi^2 + \eta^2)\left\{\frac{\partial}{\partial \xi}\left(\frac{1}{r}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{r}\frac{\partial}{\partial \eta}\right)\right\}, \tag{2'}$$

while conditions (3) to (6) are equivalent to, respectively,

$$\Psi^{\mathrm{I}} = -\frac{1}{2}r^2, \quad \frac{\partial \Psi^{\mathrm{I}}}{\partial \xi} = -\frac{\partial}{\partial \xi}\frac{1}{2}r^2 \qquad \text{at} \quad \xi = \alpha, \tag{3'}$$

$$u_\eta^{\mathrm{I}} = u_\eta^{\mathrm{II}} = 0 \qquad \text{at} \quad \xi = 0, \tag{4'}$$

$$u_\xi^{\mathrm{I}} = u_\xi^{\mathrm{II}} \qquad \text{at} \quad \xi = 0, \tag{5'}$$

and

$$u_\xi^{\mathrm{I,II}} = -\beta(p^{\mathrm{I}} - p^{\mathrm{II}}) \qquad \text{at} \quad \xi = 0. \tag{6'}$$

Note that u_η and u_z coincide with u_r and u_z only at $\xi = 0$. The requirement that the velocity vanishes at infinity should be satisfied at $\eta \to 0$ and $\xi \to 0$ simultaneously.

3. Reduction to ODE's

The general solutions of (1) which vanish at infinity are

$$\Psi^{\mathrm{I}} = (\xi^2 + \eta^2)^{-\frac{3}{2}} \int_0^\infty \lambda\eta\, G^{\mathrm{I}}(\lambda, \xi) J_1(\lambda\eta)\, d\lambda \tag{9}$$

and

$$\Psi^{\mathrm{II}} = (\xi^2 + \eta^2)^{-\frac{3}{2}} \int_0^\infty \lambda\eta\, G^{\mathrm{II}}(\lambda, \xi) J_1(\lambda\eta)\, d\lambda, \tag{10}$$

where

$$G^{\mathrm{I}} = A(\lambda)\cosh\lambda\xi + B(\lambda)\sinh\lambda\xi + C(\lambda)\lambda\xi\cosh\lambda\xi + D(\lambda)\lambda\xi\sinh\lambda\xi \tag{11}$$

and

$$G^{\mathrm{II}} = E(\lambda)e^{\lambda\xi} + F(\lambda)\lambda\xi e^{\lambda\xi}. \tag{12}$$

G^{II} is of this degenerate form to ensure that Ψ^{II} remains finite near the origin ($\xi \to -\infty$). Note that $\xi \to \infty$ is a surface within the sphere and therefore outside the domain. $A(\lambda)$ to $F(\lambda)$ are six coefficients, all functions of the continuous eigenvalue λ, to be evaluated via the use of conditions (3′) to (6′).

Application of (4′) and (5′) yields the three simple relations

$$B(\lambda) + C(\lambda) = 0 \,, \tag{13}$$

$$E(\lambda) + F(\lambda) = 0 \,, \tag{14}$$

$$A(\lambda) - E(\lambda) = 0 \,. \tag{15}$$

Here we have used the definitions

$$u_\eta = -\frac{(\xi^2+\eta^2)}{r}\frac{\partial\Psi}{\partial\xi}\,, \qquad u_\xi = \frac{(\xi^2+\eta^2)}{r}\frac{\partial\Psi}{\partial\eta}\,.$$

Conditions (3′) become

$$\int_0^\infty \lambda\eta\, G^{\mathrm{I}}(\lambda,\alpha) J_1(\lambda\eta) d\lambda = -\frac{1}{2}\frac{\eta^2}{(\alpha^2+\eta^2)^{1/2}} \tag{16}$$

and

$$\int_0^\infty \lambda\eta \frac{dG^{\mathrm{I}}}{d\xi}(\lambda,\alpha) J_1(\lambda\eta) d\lambda = \frac{1}{2}\frac{\eta\,\alpha}{(\alpha^2+\eta^2)^{3/2}}\,. \tag{17}$$

Using the Hankel transform, we find that (16) and (17) become

$$G^{\mathrm{I}}(\lambda,\alpha) = -\frac{1}{2}\frac{\alpha}{\lambda}\left(1+\frac{1}{\lambda\alpha}\right)e^{-\lambda\alpha}\,, \tag{18}$$

$$\frac{dG^{\mathrm{I}}}{d\xi}(\lambda,\alpha) = \frac{1}{2}\frac{\alpha}{\lambda}e^{-\lambda\alpha}\,. \tag{19}$$

Upon combining (13), (14), (15), (18) and (19), four of the function coefficients can be eliminated resulting in the algebraic relation

$$\begin{aligned} A(\lambda)\left[\frac{1}{\omega\,\mathrm{tgh}\,\omega} - \frac{\mathrm{tgh}\,\omega}{\mathrm{tgh}\,\omega+\omega}\right] + B(\lambda)\left[\frac{\mathrm{tgh}\,\omega-\omega}{\omega\,\mathrm{tgh}\,\omega} + \frac{\omega\,\mathrm{tgh}\,\omega}{\mathrm{tgh}\,\omega+\omega}\right] \\ = -\frac{1}{2}\left(\frac{\alpha^2}{\omega^2\sinh\omega}\left(1+\frac{1}{\omega}\right) + \frac{\alpha^2}{\omega(\sinh\omega+\omega\cosh\omega)}\right)e^{-\omega} \end{aligned} \tag{20}$$

where $\omega = \lambda\alpha$.

The most tedious part of the solution is the transformation of condition (6′). The harmonic pressure is connected to the stream function via the Cauchy-Riemann relations, viz,

$$\frac{\partial p}{\partial\eta} = -\frac{1}{r}\frac{\partial}{\partial\xi}E^2\Psi\,. \tag{21}$$

Equation (6′) is then replaced by

$$\frac{\partial}{\partial\eta}\frac{(\xi^2+\eta^2)^2}{\eta}\frac{\partial\Psi^{\mathrm{I}}}{\partial\eta}=\beta\left(\frac{1}{r}\frac{\partial}{\partial\xi}E^2\Psi^{\mathrm{I}}-\frac{1}{r}\frac{\partial}{\partial\xi}E^2\Psi^{\mathrm{II}}\right)\quad\text{at}\quad\xi=0. \tag{22}$$

After a great deal of manipulations, using the differential equation for J_1 and integration by parts, and applying the Hankel transform, we arrive at the following fourth-order ordinary differential equation

$$\beta\left(2\lambda^4\frac{d^4}{d\lambda^4}+28\lambda^3\frac{d^3}{d\lambda^3}+102\lambda^2\frac{d^2}{d\lambda^2}+90\lambda\frac{d}{d\lambda}\right)(B-A)$$
$$=\lambda^3\frac{d^2A}{d\lambda^2}+7\lambda^2\frac{dA}{d\lambda}+8\lambda A. \tag{23}$$

Boundary conditions for (23) stem from physical reasoning, the structure of the solution (9) and the constrains on the integrations by parts in the procedure to obtain (23) from (22). Thus, $\lambda^2A(\lambda)$ and $\lambda^2B(\lambda)$ must be integrable as $\lambda\to 0$ while A and B decay exponentially as $\lambda\to\infty$.

Equations (23) and (20) together with the boundary conditions outlined above provide now the final set for the solution of the original problem.

4. The force on the sphere

Before proceeding to a solution of equation (23) we discuss the force on the sphere which, presumably, is the most interesting functional of this problem. As indicated in the introduction, it is well-known (Brenner [2]) that a finite force cannot result in a departure of a sphere from an impermeable solid plane with velocity V. Conversely, a sphere approaching a plane on which no-slip conditions exist must have zero velocity at contact. These result from properties of solutions to the Stokes equations which are single-valued at the origin and elsewhere. In our case, with fluid allowed to penetrate into the membrane, a non-zero departure velocity, under the influence of a finite force, is realized with the single-valuedness of the velocity field still unaltered.

The force on the sphere can be calculated using Stimson and Jeffery's [1] formula which reduces here to

$$\frac{F_s}{6\pi\mu Va}=\frac{1}{3}\int_0^\infty\lambda^2(A(\lambda)+B(\lambda))d\lambda. \tag{24}$$

Clearly, because of the conditions associated with (23), F_s is finite as long as $\beta>0$. Indeed, Goren [6], solving for the force acting on a sphere approaching a membrane, found that the force does not increase indefinitely as the gap separating the sphere and the membrane surfaces diminishes as long as β is not zero. At high permeabilities he suggested further that maximal force is not at contact. Using spherical bipolar coordinates, Goren was unable to extrapolate his results to contact geometry and to prove his results beyond doubt, but had to resort to lubrication-theory approximations. In the next sections we shall demonstrate that Goren's conjecture is correct and obtain rigorously the asymptotic results for the cases $\beta\to\infty$ and $\beta\to 0$.

5. A sphere in contact with a highly permeable membrane, $\beta >> 1$

Assume a regular perturbation with $A(\lambda)$ and $B(\lambda)$ expanded in the forms

$$\begin{aligned} A(\lambda) &= A_0(\lambda) + \frac{1}{\beta} A_1(\lambda) + O(\frac{1}{\beta^2}), \\ B(\lambda) &= B_0(\lambda) + \frac{1}{\beta} B_1(\lambda) + O(\frac{1}{\beta^2}). \end{aligned} \tag{25}$$

Substitution of (25) in (23) shows that to $O(1)$ the differential equation reduces to

$$\left(2\lambda^4 \frac{d^4}{d\lambda^4} + 28\lambda^3 \frac{d^3}{d\lambda^3} + 102\lambda^2 \frac{d^2}{d\lambda^2} + 90\lambda \frac{d}{d\lambda}\right)(B_0 - A_0) \equiv \mathscr{L}_\lambda (B_0 - A_0) = 0. \tag{26}$$

The homogeneous solutions of (26) are

$$1, \lambda^{-2}, \lambda^{-2}\ln\lambda, \text{ and } \lambda^{-4}. \tag{27}$$

Hence, in view of the exponential decay of A and B, we obtain

$$B_0 - A_0 = 0, \tag{28}$$

where, by using (20),

$$B_0 = A_0 = \frac{1}{2}\alpha^2 \frac{1 + \dfrac{1}{\omega} + \dfrac{1}{2\omega^2}(1 - e^{-2\omega})}{\omega - \omega^2 - \dfrac{1}{2}(1 - e^{2\omega})}, \quad \omega = \lambda\alpha. \tag{29}$$

The forces can now be evaluated, with $\alpha = (2a)^{-1}$,

$$\frac{|F_s|}{6\pi\mu Va} = \frac{2}{3}\int_0^\infty \lambda^2 B_0(\lambda) d\lambda = 1.0722 + O(\frac{1}{\beta}). \tag{30}$$

F_s agrees exactly with Goren's evaluations for small gaps at the limit $\beta \to \infty$.

The equation for the first order perturbation is now of the form

$$\mathscr{L}_\lambda (B_1 - A_1) = \lambda\left(\lambda^2 \frac{d^2}{d\lambda^2} + 7\lambda \frac{d}{d\lambda} + 8\right) A_0. \tag{31}$$

Since

$$A_0 = -\frac{1}{2}\frac{\alpha^2}{\omega^2}\left(1 + \frac{\omega^3}{6} + O(\omega^5)\right) \quad \text{as} \quad \lambda \to 0, \tag{32}$$

while A_0 decays exponentially as $\lambda \to \infty$, $B_1 - A_1$ must be of $O(\lambda^2)$ as $\lambda \to 0$ and decay exponentially at infinity. By (24) we have

$$\frac{|F_s|}{6\pi\mu Va} - 1.0722 = \frac{1}{3\beta}\int_0^\infty \lambda^2 (A_1 + B_1) d\lambda + O(\frac{1}{\beta^2}). \tag{33}$$

6. The case of low permeability, $\beta \ll 1$

When the permeability of the membrane is low and β is small, as in most practical physical realizations, the solution to equation (23) becomes singular. There exists a boundary layer in the eigenvalue domain near $\lambda = 0$ and, therefore, we must resort to singular-perturbation techniques to obtain the functions coefficients in the form of matched asymptotic expansions.

The outer region

Assume $A(\lambda)$ and $B(\lambda)$ of the form

$$A(\lambda) = \sum_{n=0}^{\infty} f_n(\beta) A_n(\lambda), \quad B(\lambda) = \sum_{n=0}^{\infty} g_n(\beta) A_n(\lambda), \tag{34}$$

where $f_0 = g_0 = 1$ and $f_{n+1} = o(f_n), g_{n+1} = o(g_n)$. From equation (23) we obtain

$$\lambda^2 \frac{d^2 A_0}{d\lambda^2} + 7\lambda \frac{dA_0}{d\lambda} + 8A_0 = 0 \tag{35}$$

with the homogeneous solutions $\lambda^{-2}, \lambda^{-4}$. Hence to this order, $A_0 = 0$ and B_0 is evaluated using (20). Obviously, B_0 decays exponentially as $\lambda \to \infty$, however,

$$B_0 = -\frac{\alpha^2}{2} \cdot \frac{1 + \dfrac{1}{\omega} + \dfrac{1}{2\omega^2}(1 - e^{-2\omega})}{\sinh^2\omega - \omega^2} \to -\frac{24}{\lambda^5} \text{ as } \lambda \to 0, \ (\alpha = \frac{1}{2}). \tag{36}$$

Thus, a thin boundary layer exists near $\lambda = 0$. The solution within this layer should be matched to (36).

The inner region

Consider expansions of the form

$$A(\nu) = \sum_{n=0}^{\infty} F_n(\beta)\hat{A}_n(\nu), \quad B(\nu) = \sum_{n=0}^{\infty} G_n(\beta)\hat{B}_n(\nu) \tag{37}$$

where $\nu = \lambda/\beta^{\ell}$. Expanding for small β and balancing the appropriate terms in (20) and (23) we find that $\ell = \frac{1}{4}$, $F_0 = \beta^{-1/2}$ and $G_0 = \beta^{-5/4}$. The equations for $\hat{A}_0(\nu)$ and $\hat{B}_0(\nu)$ now become

$$\hat{A}_0 + \frac{\nu^3}{48}\hat{B}_0 = -\frac{1}{2\nu^2} \tag{38}$$

and

$$\mathscr{L}_\nu(\hat{B}_0 - \hat{A}_0) = \nu^3 \frac{d^2\hat{A}_0}{d\nu^2} + 7\nu^2 \frac{d\hat{A}_0}{d\nu} + 8\nu\hat{A}_0 \tag{39}$$

where for convenience we have set $\alpha = \frac{1}{2}$ without loss of generality. $\hat{A}_0$ and $\hat{B}_0$ must satisfy the boundary conditions at $\lambda \to 0$ $(\nu \to 0)$ and match the outer solution (36) as $\nu \to \infty$.
Substituting (38) into (39) yields

$$\left\{2\nu^4 \frac{d^4}{d\nu^4} + 28\nu^3 \frac{d^3}{d\nu^3} + (102\nu^2 + \frac{\nu^6}{48}) \frac{d^2}{d\nu^2} + (90\nu + \frac{13\nu^5}{48}) \frac{d}{d\nu} + \frac{35}{48}\nu^4\right\}\hat{B}_0 = 0. \tag{40}$$

It is useful to study the behaviour of the solutions of (40) at $\nu \to 0$ and $\nu \to \infty$. In the former case we have as before

$$1, \nu^{-2}, \quad \nu^{-2}\ln\nu \quad \text{and} \quad \nu^{-4}, \tag{41}$$

where the last solution is not permissible due to the conditions at $\nu \to 0$. As $\nu \to \infty$ we observe that

$$\hat{B}_{0_{\rm I}} = \nu^{-5},\ \hat{B}_{0_{\rm II}} = \nu^{-7}, \ \text{and} \ \frac{d^2}{d\nu^2}\hat{B}_{0_{\rm III,IV}} \sim \frac{1}{\nu}\exp(\pm i\frac{\nu}{\sqrt{48}}) \tag{42}$$

where obviously the diverging oscillatory solutions $\hat{B}_{0_{\rm III}}$ and $\hat{B}_{0_{\rm IV}}$ are not appropriate.

The general inner solution, to this order, which matches the outer solution is of the form

$$\hat{B}_0 = -24\hat{B}_{0_{\rm I}} + C_2\hat{B}_{0_{\rm II}}. \tag{43}$$

Unfortunately, this form does not enable a unique determination of $\hat{B}_0$. Uniqueness can nevertheless be secured in several ways as outlined in the Appendix. Once $\hat{B}_0$ is obtained uniquely the forces can be evaluated by constructing a composite expansion which is uniformly valid throughout the domain.

The force

It is clear that, to this degree of approximation, the expression for the force acting on the sphere will not involve integration over $A(\lambda)$ since $B(\lambda)$ is $O(\beta^{-5/4})$ but $A(\lambda)$ is $O(\beta^{-1/2})$, and, therefore, negligible. Construction of a composite expansion yields

$$\frac{F_s}{6\pi\mu Va} = \frac{1}{3}\int_0^\infty \lambda^2\{F_0(\beta)\hat{B}_0(\nu) + \hat{B}_0(\lambda) + \frac{24}{\lambda^5}\}d\lambda + O(1). \tag{44}$$

The combination of the last two terms in the integral is integrable but $O(1)$ and thus can be ignored to this order of magnitude. Hence, with $\lambda = \nu\beta$,

$$\frac{|F_s|}{6\pi\mu Va} = \frac{1}{3\beta^{1/2}}\left|\int_0^\infty \nu^2\hat{B}_0(\nu)d\nu\right| + O(1) = \frac{0.816}{\beta^{1/2}} + O(1). \tag{45}$$

This result confirms Goren's [6] coefficient $(2/3)^{1/2}$ obtained from lubrications approximation and validates his calculations for small gaps at large β^{-1}.

7. Concluding remarks

When considering the flow around a sphere departing from a permeable membrane and the force acting on it, the tangent-sphere coordinates enable a reduction of the Stokes equations to a fourth-order ordinary differential equation. This equation can be solved numerically for any permeability or through the use of regular and singular perturbation for the asymptotic cases $\beta >> 1$ and $\beta << 1$, respectively. Our results at contact confirm and supplement the results obtained by Goren [6] for the case of a sphere approaching a thin permeable membrane with finite separation. It is now clear that at high permeabilities the maximum force acting on the sphere is not at contact.

Appendix

One route to obtain the correct combination of $\hat{B}_{0_{\rm I}}$ and $\hat{B}_{0_{\rm II}}$ in (43) is as follows: Suppose that we can find the exact combinations of the solutions near $\nu \to 0$ (41) associated with each of the solutions $\hat{B}_{0_{\rm I}}$ and $\hat{B}_{0_{\rm II}}$. Since the first three are permissible it is sufficient to choose a combination of $\hat{B}_{0_{\rm I}}$ and $\hat{B}_{0_{\rm II}}$ which eliminates the fourth singular one. This procedure, however, is not obvious since the ability to analytically match solutions over the semi-infinite domain of ν is rarely possible (Dingle [12]).

Another possibility is via the definition of the force. Multiplying equation (40) by ν^2 and integrating over the domain using integration by parts, we obtain

$$\int_0^\infty \nu^2 \hat{B}_0(\nu) d\nu = \lim_{\nu \to \infty} \frac{\nu^8 \dfrac{d\hat{B}_0}{d\nu} + 5\nu^7 \hat{B}_0}{288}. \tag{A-1}$$

Note that, since

$$B_{0_{\rm I}} = -\frac{24}{\nu^5} + O(\nu^{-9})$$

and

$$B_{0_{\rm II}} = \frac{C_2}{\nu^7} + O(\nu^{-11}) \tag{A-2}$$

as $\nu \to \infty$, the RHS of (A-1) simply becomes $-C_2/144$. Comparing this with a direct integration of (43) we get

$$C_2 = \frac{24 \displaystyle\int_0^\infty \nu^2 \hat{B}_{0_{\rm I}} d\nu}{\dfrac{1}{144} + \displaystyle\int_0^\infty \nu^2 \hat{B}_{0_{\rm II}} d\nu}. \tag{A-3}$$

The numerical procedure which we have adopted inherently uses the first of the above methods. The results were then checked to ensure that (A-3) is satisfied. Substituting $y = \nu^3 \hat{B}_0$ we find that equation (40) is transformed to

$$\left\{2\nu^4 \frac{d^4}{d\nu^4} + 4\nu^3 \frac{d^3}{d\nu^3} + \left(-6\nu^2 + \frac{\nu^6}{48}\right)\frac{d^2}{d\nu^2} + \left(6\nu + \frac{7\nu^5}{48}\right)\frac{d}{d\nu} + \left(-6 + \frac{8\nu^4}{48}\right)\right\} y = 0\,, \tag{A-4}$$

where (41) and (42) become, respectively,

$$\nu^3,\ \nu,\ \nu \ln \nu \ \text{ and } \ \nu^{-1}\,, \tag{A-5}$$

$$\nu^{-2},\ \nu^{-4} \ \text{ and } \ \frac{d^2 y}{d\nu^2} \sim \frac{1}{\nu}\exp\left(\pm i\,\frac{\nu}{\sqrt{48}}\right). \tag{A-6}$$

Equation (A-4) is then integrated using a 5-diagonal Gauss elimination procedure requiring that $y(0) = 0$ and y decay as $24/\nu^2$ as $\nu \to \infty$. The limit of $\nu^4(y - 24\nu^{-2})$ as $\nu \to \infty$ is then checked and compared with the value of C_2 obtained from the force.

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